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# Stability Analysis of Two Cooperating Robot Manipulators

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## Abstract

This work is a general, nonlinear approach (Small Gain Theorem) in stability analysis of two robot manipulators with unstructured dynamic models. The interaction between the robot and the environment is a special case of the general analysis. This stability condition confirms the simplified results in the stability analysis of the linearly treated robots.

The stability analysis has been investigated using unstructured models for dynamic behavior of robot manipulators. This unified approach of modeling robot dynamics is expressed in terms of sensitivity functions. It allows us to incorporate the dynamic behavior of all the elements of a robot manipulator (i.e. actuators, sensors and the links structural compliance) in addition to the rigid body dynamics.

## 1. Introduction

One of the significant issues in robot motion control is the stability analysis of the interaction between two manipulators or of the manipulator and the environment. Robotic deburring and grinding are practical examples of the interaction of the robot with the environment [5]. In this analysis, the interaction between two robots will be analyzed in detail. We propose a unified approach to model the dynamic behavior of a robot manipulator based on an input/output functional relationship. This unified approach of modeling robot dynamics allows us to incorporate both the dynamic behavior of all the elements of a robot manipulator and the rigid body dynamics. We are looking for a dynamic model that can represent the complete dynamic behavior of any robot in a very general form. We believe that there may be enough components in the robot arm so that rigid body dynamics is not sufficient for modeling. In fact, in many hydraulic robots, the actuators and the servovalves dynamics dominate the total dynamic behavior of the robot.

We try to avoid using structured dynamic models such as first or second order transfer functions as general representation of the dynamic behavior of the components of the system (e.g. servovalves in the hydraulic robots and the gear stiffness in the non-direct drive systems). Extending this idea to modeling the environment, we also avoid using mass and spring systems to describe the dynamic behavior of the environment. These models are not general and the stability analysis consequently results in non-general academic conclusions (illusions). References 9 and 10 contain some basic issues in general dynamic model for the environment.

## 2. Dynamic Model of a Robot with Tracking Capability

We define an *ideal* tracking robot as that system which: 1) is able to follow all trajectories and, 2) rejects all the disturbances under all circumstances. The above two conditions can be integrated to define an ideal tracking robot as a dynamical system which is able to exactly follow any commanded trajectory as long as it does not interact with an infinitely stiff environment. Let us define the sensitivity function as a mapping from the disturbances to the motion of the robot. Thus the sensitivity function of an ideal tracking manipulator always results in zero deviation of the manipulator's trajectory for all bounded disturbances. In the linear domain, one can consider a robot with infinite bandwidth and zero sensitivity transfer function matrix as an example of an ideal robot. A physical system with such characteristics is modeled as an ideal source of flow (current source)(14). The dynamic behavior of an ideal one dimensional tracking robot can be represented by an ideal current source as shown in Figure 1a. $I_o$ is an analogy of the commanded trajectory, while v, the imposed voltage, represents the imposed disturbance or the force on the robot. An ideal current source provides a current, $I_o$, which could be any arbitrary time function but independent of the magnitude of v.

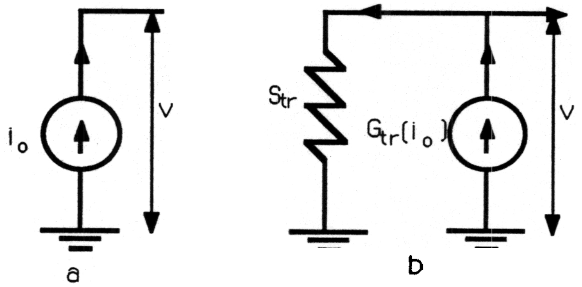


Figure 1: Electrical Model of a System with Tracking Capability

In reality, no robot behaves as an ideal tracking robot. Robots can track only those arbitrary trajectories and reject those disturbances that contain components with bounded frequency ranges and magnitudes. This is from the limitation in the bandwidths and the magnitude of the power of the controlled systems. (Noise reduction and robustness to high frequency unmodeled dynamics are the primary reasons for limitation of the controlled systems' bandwidth (1,6)). We define these robots as *average* tracking robots. To arrive at a scalar model for a robot with average tracking capability, we propose a limited bandwidth source of flow in parallel with a nonlinear resistor (Figure 1b). The available current at the output, given by Equation 1, is the algebraic addition of the current from the resistor and the actual current generated by the source.

$$= G_{tr}(I_o) - S_{tr}(v) \qquad (1)$$

The mapping, $S_{tr}$, represents the sensitivity of the tracking system, while $G_{tr}$ represents the dynamics of the source of flow. $S_{tr}$ is, in the general case, any combination of linear and/or nonlinear electrical components (e.g. resistors, inductors, capacitors). If a linear resistor is being used, then $S_{tr}=1/R$ where R is the electrical resistance measured in Ohms. $I_o$ represents the commanded current while v represents the interaction force which is determined by the interacting system. Note that $G_{tr}(I_o)$ characterizes the actual current from the source of flow which may be different from $I_o$.

In the ideal model, the actual current from the source of flow is equal to the commanded flow in the current source. In a realistic model, however, the internal dynamics of the source of flow is operating on the commanded current to produce $G_{tr}(I_o)$. In fact, an ideal linear tracking robot can mathematically be defined as that system with $G_{tr}=I_n$ and $S_{tr}=0_n$ where n is the degree of freedom of the robot. For realistic models, $G_{tr}$ is approximately equal to the identity mapping only for some bounded frequency ranges. If $S_{tr}$ is such that for any bounded value of v, the current in the resistor is zero (infinite R in the linear case), then only $G_{tr}(I_o)$, the current from the source of flow, will be available at the output. This indicates that $S_{tr}$ is a measure of how "good" a tracking system is. On the other hand, a very small voltage across the resistor will cause the system to deviate from its trajectory if $S_{tr}$ has a "large" amplification (small R in the linear case). Manipulators with such characteristics are defined as *weak* tracking robots. The rigorous definition from the stand point of the nature of $S_{tr}$ for weak tracking robots will be given later.

We assume that $I_o$, I and, v, belong to any Banach (normed space) or Hilbert space (2) and $S_{tr}$ and $G_{tr}$ are nonlinear operators that map the particular space used onto itself. An ideal tracking robot has a zero gain for $S_{tr}$ while the average tracking robot posesses a finite gain for $S_{tr}$. (The gain of an operator is defined in Appendix A). The block diagram in Figure 2 represents the dynamic behavior of the system shown in Figure 1b.

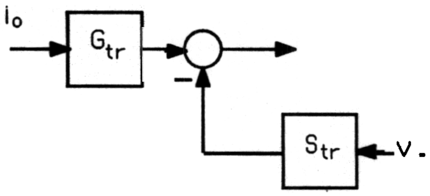


Figure 2: Nonlinear Block Diagram of a Multi-Degree of Freedom Robot with Tracking Capability

Note that although the circuit of Figure 1 is inadequate to represent a multi-degree of freedom robot, the block diagram of Figure 2 and equation 1 are general enough to cover the nonlinear dynamic behavior of the multi-degree of freedom robots with tracking controllers. For multi-degree of freedom robots, $I_0$, I, V are vectors. Reference 11 describes the dynamic behavior of an active end-effector with the above method.

## 3. Dynamic Model of a Robot of a Robot with Force Control Capability

Since the dynamic analysis of a robot with tracking capability is the dual to the dynamic analysis of a robot with force control capability, we will arrive at a dynamic model of the latter in a fashion similar to the former. Using that reasoning, a tracking robot cannot be viewed buried in an infinitely stiff environment, a force control robot cannot be viewed in a free (vacuum) environment. The definition of force control for a robot is meaningless if the robot is not constrained at least in one direction. From here on after, "force control robot", implies the control of force in a space (in directions) in which the robot is constrained; however the robot may or may not be constrained in the remaining directions (12,13,20,21).

We define an *ideal* force control robot (again in a particular constrained space) as a system that: 1) is able to follow all commanded forces and, 2) rejects all the trajectory disturbances under all circumstances. The above two conditions can be integrated to define an ideal force control robot as that system which is able to exactly follow all commanded forces as long as it is not in a free environment (an environment with zero stiffness). An ideal force control robot will impose the commanded force even if the interacting system (a robot or an environment) is "trying to escape from it". The ideal force control robot can spend any amount of power in order to impose the commanded force onto another system. Again, one can think of a robot with infinite bandwidth and zero sensitivity as an ideal force control manipulator. The sensitivity, in this case, is defined as a mapping from trajectory disturbances to the contact forces. We model an ideal force control robot with a source of effort (e.g. a voltage source) as shown in Figure 3a (31). However, any realistic model of a force control robot in a constrained space has a non-zero sensitivity. These are defined as the *average* force control robots. A proper model of a one degree of freedom force control system in a constrained direction is proposed by Figure 3b.

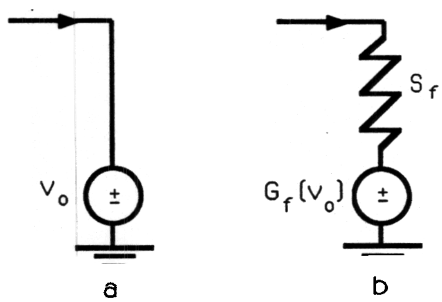


Figure 3: Electrical Model of a System with Force Control Capability

The constitutive equation of the model is:

$$V = G_f(V_0) + S_f(I) \qquad (2)$$

where $S_f$ represents the sensitivity of the force control system while $G_f$ represents the dynamics of the source of effort. Although the circuit of Figure 3 is inadequate to simulate a multi-degree of freedom robot, equation 2 is general enough to cover the nonlinear dynamic behavior of the multi-degree of freedom force control capability of a constrained robot. As in the previous model, $S_f$ is described by a generalized resistance. In the linear case, $S_f$ is characterized with the resistance R. $G_t(V_0)$ is prescribed by the source of effort while the current, I is determined by the interacting system. $S_f$ is a measure of how "good" the force control system is. The smaller the amplification of $S_f$ is (the short circuit in the limiting case) the better the force control capability will be. A robot with "good" force control capability (small R in the linear case) will exert the commanded force onto the interacting system independent of its imposed trajectory. An ideal force control robot has a zero gain for $S_f$ while the average force control robot posesses a finite gain for $S_f$. (the gain of an operator is defined in Appendix A). A robot with poor force control capability (large R in the linear case) needs a very stiff environment (i.e. an environment that cannot be moved) to follow the commanded force. Manipulators with such characteristics are defined as *weak* force control robots. The rigorous definition for the weak force control capability from the stand point of nature of $S_f$ is given in Section 6.2. The rigorous definition from the stand point of the nature of $S_{tr}$ for weak tracking robots can be arrived at in a similar fashion.

We assume that $S_f$ and $G_f$ are nonlinear operators that map Banach spaces. The gain of the operator $S_f$ is zero and non-zero finite scalars for the ideal and the average force control robots respectively. The block diagram in Figure 4 shows a realistic dynamic representation of a multivariable nonlinear force control capability of a robot in a constrained space via the nonlinear operator domain.

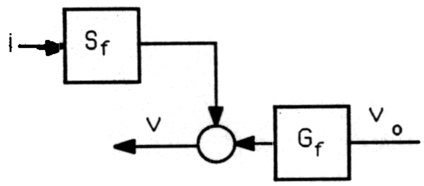


Figure 4: Nonlinear Block Diagram of a Multi-Degree of Freedom Force Control Capability of a Robot

## 4. Interaction of Two Robots

The objective is to arrive at the stability condition for the interaction of two robot manipulators. The stability criteria for the interaction of a robot manipulator and the environment is a particular case of this general analysis.

From a merely physical perspective, any two interacting robots must complement each other. Along any of the directions in which the interaction takes place, if a robot's dynamic behavior is governed by an admittance, the other's dynamic behavior must be governed by an impedance (3,4,7,8,10,11,15). The impedance is defined as an operator with a flow variable (current, velocity) as its input and an effort variable (voltage,force) as its output. The admittance is defined as an operator with an effort variable as its input and a flow variable as its output. Considering two interacting robots, the tracking robot accepts a flow (trajectory) as a command and reflects an effort (force) as the output (impedance), while the force control robot accepts an effort as a command and reflects a flow as the output (admittance).

We start with the stability analysis of two one-dimensional robots. The interacting robots are denoted as the TR-robot (tracking robot) and the F-robot (force control robot). Figure 5 represents the analog circuit for the interaction of two one-dimensional robot manipulators.

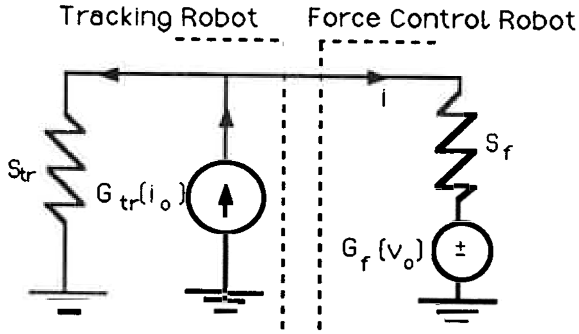


Figure 5: The Analogue Circuit for the Interaction of two Robots in a Particular Direction

The linear time invariant approach is first used to arrive at stability criteria of the interaction of the two one-degree-of-freedom linearly-treated robots. These concepts will then be extended, in Section 6, to the nonlinear domain for interactions that take place along more than one degree of freedom. The interaction between two robots in the nonlinear operator domain is represented in Figure 6. Note that, although the circuit of Figure 6 is inadequate to simulate the interaction of two multi-degree of freedom robots, the block diagram of Figure 6 is general enough to cover the nonlinear dynamic behavior of the interaction of two multi-degree of freedom robots.

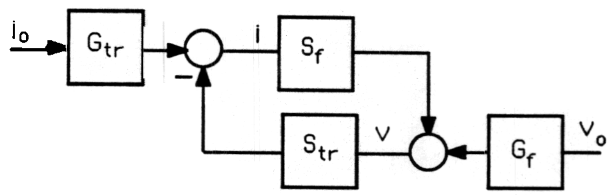


Figure 6: The Operator Diagram for the Interaction of the Robots (Combination of Figures 3 and 4)

## 5. Interaction of two One-Degree-of-Freedom Robots in the Linear Time Invariant Domain

The objective of this section is to show some concepts on the interaction (in particular the stability criteria) of two one-dimensional, linearly-treated robot manipulators. The following analysis refers to Figure 5 where all the parameters of the analog circuit are assumed to be linear. The current, i, represents the actual trajectory of the TR-robot while the voltage, v, is the actual force exerted by the F-robot. The explicit relationships of i and v in terms of the parameters of the circuit are given by equations 3 and 4.

$$= \frac{R_{tr}\, G_{tr}\, i_o - G_f v_o}{R_{tr} + R_f} \qquad (3)$$

$$v = \frac{(G_f v_o + R_f\, G_{tr}\, i_o)\, R_{tr}}{R_{tr} + R_f} \qquad (4)$$

Sections 5.1 and 5.2 describe the stability conditions of two linearly-treated one-degree-of-freedom robots. Section 5.1 describes the stability analysis when both robots are dynamically considered ideal. In Section 5.2 the stability of the ideal tracking robot with a weak force control robot will be investigated.

### 5.1 Interaction of an Ideal TR-robot with an Ideal F-robot

To simulate the interaction between an ideal linear TR-robot and an ideal linear F-robot, we let the value of $R_{tr}$ tend to infinity while that of $R_f$ to zero. ($R_{tr} \rightarrow \infty$ and $R_f \rightarrow 0$ imply zero sensitivity for both robots.) Under these limiting conditions, the current, i, and the voltage, v, tend to $G_{tr} i_o$ and $G_f v_o$ respectively. Moreover, for ideal robots, both $G_{tr}$ and $G_f$ are unity. Since i and v are bounded values (for a given set of bounded $i_o$ and $v_o$), the interaction under these particular conditions results in a very stable overall system. While the ideal TR-robot moves along the desired trajectory, it will not specify any bound on a possible interaction force. In contrast, the ideal F-robot imposes the desired force onto the TR-robot without specifying any trajectory. These two robots physically complement each other and their interaction is very stable.

### 5.2 Interaction of an Ideal TR-robot with Weak F-robot

In this case the simulation is accomplished by letting the values of both $R_{tr}$ and $R_f$ tend to infinity. $R_f \rightarrow \infty$ simulates an infinite sensitivity which implies a poor force control property in the linear time invariant domain. It is clear that under these limiting conditions the voltage v approaches infinity. While the TR-robot tries to impose the trajectory given by its commanded input onto the F-robot, the latter will hardly be moved. Thus a very large force will be generated during the interaction. The F-robot has lost all its force control capability, now having tracking capability. In other words, the F-robot resembles an ideal source of current with zero current (e.g. "infinitely" stiff environment). The interaction under these limiting conditions approaches the interaction between two ideal tracking robots. These robots will not physically complement each other; instability occurs and the contact force is unbounded. The results of this stability analysis confirms the results given in references; the interaction of an ideal tracking robot with an infinitely stiff environment results in an unstable

## 6. Interaction of the Robots in the Nonlinear Domain

The model of the interaction of two robots is represented by Figure 6 where equations 1 and 2 describe the governing nonlinear laws of the interaction. The current, i, represents the actual trajectory of the TR-robot while the voltage, v, is the actual force exerted by the F-robot. Note that i and v are vectors of a proper dimension that represents the space in which two robots interact.

We will first analyze the two types of interactions described in Sections 5.1 and 5.2, arriving at proper multi-degree-of-freedom and nonlinear generalization of the results obtained for the linear case. The analysis will then be extended to cover other types of interactions.

### 6.1 Interaction of an Ideal TR-robot with an Ideal F-robot

To simulate these ideal robots the gains (i.e. amplification) of the operators $S_{tr}$ and $S_f$ are set to zero. Applying the Small Gain theorem, under the condition on $S_{tr}$ and $S_f$, the following bounds on v and are derived in Appendix B:

$$||i||_p \leqslant \kappa_1 ||i_o||_p \qquad (5)$$

$$||v||_p \leqslant \kappa_2 ||v_o||_p \qquad (6)$$

where $\kappa_1$ and $\kappa_2$ are positive finite scalars defined as the gains of $G_{tr}$ and $G_f$ respectively. In fact, for the ideal robots, both $\kappa_1$ and $\kappa_2$ are unity. Therefore, if the input signals $i_o$ and $v_o$ are assumed to be bounded in the sense that they belong to the $L_p$ space, then v and i will be bounded and they will also belong to the $L_p$ space. Moreover, it is shown in Appendix B that the mappings $(i_o,v_o) \mapsto i$ and $(i_o,v_o) \mapsto v$ are linearly bounded in the sense of Definition 5 of Appendix A. Thus the system shown in Figure 6 is $L_p$-stable. Note that this is the same result as the one obtained in the linear case.

A physical interpretation of this result can be given if space $L_2$ is used. (The norm of a signal that belongs to the $L_2$ space represents the energy of the signal). If the input signals are assumed to be of finite energy, then the power exchange during the interaction is always finite and bounded by the product of the energy of the input signals.

## 6.2 Interaction of an Ideal TR-robot with a Weak F-robot

In the linear case we modeled a weak F-robot by letting the value of $R_f$ tend to infinity. This idea can be extended to the operator domain by assuming that $S_f$ is not an $L_p$-stable operator. To state the $L_p$ instability of $S_f$, any condition required for the $L_p$ stability can be violated (see Appendix A Definition 5). We assume that the operator $S_f$ is not $L_p$-stable in the sense that it can not be linearly bounded although it may map $L^n_{pe} \mapsto L^n_{pe}$ and $L^n_p \mapsto L^n_p$. Note that this is a realistic assumption to model the nature of $S_f$ for this class of robots. The physical interpretation of this assumption is as follows: linearly increasing displacements applied at the end-point of the F-robot generate contact forces that can not be linearly bounded. This definition implies the characteristics of a poor force control robot that is very sensitive to the trajectory disturbances. A similar definition can be given for a weak tracking robot. The operator $S_{tr}$ for a weak tracking robot is not $L_p$-stable in the sense that it can not be linearly bounded although it may map $L^n_{pe} \mapsto L^n_{pe}$ and $L^n_p \mapsto L^n_p$. The definition implies that the linearly increasing force disturbances applied at the end-point of the TR-robot generate a trajectory that cannot be linearly bounded. (A poor tracking system that responds to the disturbances very well.)

In Appendix C, it is proven that under the conditions assumed for the operator $S_f$, the system on Figure 6 is not $L_p$-stable in the sense that the system can not be linearly bounded. This instability result confirms the instability result which was obtained in the linear case.

## 7. Other Types of Interactions Between Robots

So far, we have analyzed the interaction of two active systems when these take the form of an ideal TR-robot with an ideal F-robot or, an ideal TR-robot with a weak F-robot. However, the interactions of two robots are not constrained to be in these two forms. We have chosen these two types of interactions because the analysis of other types of interactions are similar to those already considered in Sections 6.1 and 6.2. The following table summarizes the possible interactions between two robots.

| | Ideal Tracking Robot | Average Tracking Robot | Weak Tracking Robot |
|---|---|---|---|
| Ideal Force Control Robot | Stable | Stable | Unstable |
| Average Control Robot | Stable | Conditionally Stable | ? |
| Weak Force Control Robot | Unstable | ? | ? |

Table1: Interaction of Two Active Systems with its Associated Stability Property

According to Table 1, there are three cases of interactions between two robots that always lead to stable interactions. The one that combines an ideal tracking robot with an ideal force control robot has been analyzed in Sections 5.1 and 6.1. The $L_p$-stability of the other two case (average force control robot and an ideal tracking robot, an average tracking robot and an ideal force control robot) can be shown by inspection of inequalities B9 and B10 in Appendix B. If the gains of the operators $S_{tr}$ and $S_f$ are finite and their product is less than 1, then the $L_p$-stability of the interaction is guaranteed; if the product is greater than 1, then nothing can be said about the stability of the interaction. For these types of interactions, either the gain of $S_f$ is finite and that of $S_{tr}$ is set to zero or the gain of $S_{tr}$ is finite and that of $S_f$ set to zero. The fact that at least one of the operators' gain is set to zero guarantees that the product of the gains of $S_f$ and $S_{tr}$ is less than 1

There are two categories of interactions that $L_p$-stability can be guaranteed (an ideal tracking robot and a weak force control robot, an ideal force control robot and a weak tracking robot). The $L_p$-instability of these two cases can be shown by inspection of inequalities E8 of Appendix C. The product of these gains is zero, since one of them is zero. The end point trajectory or the contact force is not linearly bounded by the norm of the inputs. For example, if the gain of $S_{tr}$ is zero and the gain of $S_f$ is nonlinealry bounded, the contact force will be nonlinearly bounded. On the other hand, if the gain of $S_f$ is zero and the gain of $S_{tr}$ is nonlinearly bounded, then the trajectory will be nonlinearly bounded by the norms of the inputs.

Table 1 shows one case in which the stability status of the interaction between the two robots does not have a categorical answer (i.e. unstable or stable). This case refers to the interaction of an average tracking robot with an average force control robot. The stability depends on the particular characteristics of each of the robots. Inequalities B9 and B10 of Appendix B give a sufficient condition for the stability of the interaction. If the product of the gains of the operators $S_{tr}$ and $S_f$ is less than 1, then the $L_p$-stability of the interaction is guaranteed; if the product is greater than 1, then nothing can be said about the stability of the interaction.

There are three case in the table that are represented by "?". In all these cases one or both systems are unstable operators. There is no general theorem that gives conditions for $L_p$-stability (or $L_p$-instability) for these cases. Note that $\kappa_3\ \kappa_4$ can be greater than 1 and thus one cannot arrive at inequality B9 and B10 for stability analysis. The most recent result on stability analysis of the closed loop systems is given in reference (16). This reference gives instability condition when one of the system is linear, time invariant and unstable while the other one is nonlinear, time varying and $L_2$-stable. If both systems are nonlinear time variant, then nothing can be said about the stability status of their interaction.

## 8. Conclusion

The most stable interaction occurs between an ideal tracking robot and ideal force control robot. An average tracking robot and an average force control robot behave as ideal robots when working inside their bandwidth. During manipulation, the tracking robot will take care of the trajectory while the force control robot will take care of the interacting force (the two robot physically complement each other). Note that no external control architecture is needed to accommodate the force developed during the interaction (i.e. modulate the impedance of the tracking robot). An environment can be modeled by a force control robot with a zero input command. Since a force control robot with zero input command can no longer fulfill the task of accommodating the interaction force, therefore some compliance must be developed for the tracking robot for stability.

## Appendix A

Definitions 1 to 7 will be used in the stability proof of the closed-loop system (17,18,19).

<u>Definition 1</u>: For all $p \in [1,\infty]$, we label as $L^n_p$ the set consisting of all functions $f=(f_1,f_2,\ldots,f_n)^T: [0,\infty)\to\mathbb{R}^n$ such that:

$$\int_0 |f_i|^p\, dt < \infty \qquad \text{for} \qquad 1,\ 2,\ .n$$

<u>Definition 2</u>: For all $T\in[0,\infty)$, the function $f_T$ defined by:

$$f_T = \begin{cases} f & 0\le t\le T \\ 0 & T<t \end{cases}$$

is called the truncation of f to the interval [0,T]

Definition 3: The set of all functions $f=(f_1,f_2,...,f_n)^T$: $[0,\infty) \to \mathbb{R}^n$ such that $f_T \in L^n_p$ for all finite T is denoted by $L^n_{pe}$. f by itself may or may not belong to $L^n_p$.

Definition 4: The norm on $L_p^n$ is defined by:

$$\| f \|_p \quad \left( \sum_{i=1}^{n} \| f_i \|_p^2 \right)^{1/2}$$

where $\| f_i \|_p$ is defined as:

$$\| f_i \|_p = \left( \int_0^\infty | f_i |^p \, dt \right)^{1/p}$$

Definition 5: Let $V(.)$: $L^n_{pe} \to L^n_{pe}$. We say that the operator $V(.)$ is $L_p$-stable, if:

a) $V(.)$: $L_p^n \to L_p^n$

b) there exist finite real constants $\alpha_4$ and $\beta_4$ such that:

$$\|v(e)\|_p < \alpha_4 \|e\|_p + \beta_4 \qquad \forall \; e \in L^n_p$$

According to this definition we first assume that the operator maps $L^n_{pe}$ to $L^n_{pe}$. It is clear that if one does not show that $V(.)$:$L^n_{pe} \to L^n_{pe}$, the satisfaction of condition (a) is impossible since $L^n_{pe}$ contains $L^n_p$. Once the mapping, $V(.)$, from $L^n_{pe}$ to $L^n_{pe}$ is established, then we say that the operator $V(.)$ is $L_p$-stable if, whenever the input belongs to $L^n_p$, the resulting output belong belongs to $L^n_p$. Moreover the norm of the output is no larger than $\alpha_4$ times the norm of the input plus the offset constant $\beta_4$.

Definition 6: The smallest $\alpha_4$ such that there exist a $\beta_4$ so that inequality b of Definition 5 is satisfied is called the gain of the operator $V(.)$.

Definition 7: Let $V(.)$:$L^n_{pe} \to L^n_{pe}$. The operator $V($. is said to be causal if:

$$V(e)_T = V(e_T) \quad \forall \; T<\infty \quad \text{and} \quad \forall \; e \in L^n_{pe}$$

## Appendix B

The objective is to prove the $L_p$-stability of the system given by equations 1 and 2 when the gains of $S_{tr}$ and $S_f$ are set to zero. It is assumed that the operators $G_{tr}$, $G_f$, $S_{tr}$, and $S_f$ are $L_p$-stable and causal. These two assumptions on each of the operators guarantee the existence of finite constants $\kappa_1, \kappa_2, \kappa_3, \kappa_4, \in \mathbb{R}^+$ and $\delta_1, \delta_2, \delta_3, \delta_4, \in \mathbb{R}$ such that the following inequalities are true. [2]

$$\| G_{tr} (i_o)_T \|_p \leq \kappa_1 \| i_{oT} \|_p + \delta_1 \qquad (B1)$$

$$\| G_f (v_o)_T \|_p \leq \kappa_2 \| v_{oT} \|_p + \delta_2 \qquad (B2)$$

$$\| S_{tr} (v)_T \|_p \leq \kappa_3 \| v_T \|_p + \delta_3 \qquad (B3)$$

$$\| S_f (i)_T \|_p \leq \kappa_4 \| i_T \|_p + \delta_4 \qquad (B4)$$

where $T \in \mathbb{R}^+$.

Proof: First we show that whenever $i_o$ and $v_o$ belong to $L^n_{pe}$ then v and i belong to $L^n_{pe}$ under the limiting condition on $\kappa_3$ and $\kappa_4$. It is clear that if i and $v \notin$ to $L^n_{pe}$ whenever $i_o$ and $v_o \in L^n_{pe}$ and $\kappa_3$ and $\kappa_4$ are set to zero, then i and $v \notin L^n_p$ either and the system of equations 1 and 2 will not be $L_p$-stable. The truth of equality B5 for any finite T can be observed from Figure 6.

$$i_T = G_{tr} (i_o)_T + S_{tr} (v)_T \qquad (B5)$$

Hence we have:

$$\| i_T \|_p \leq \| G_{tr} (i_o)_T \|_p + \| S_{tr} (v)_T \|_p \qquad (B6)$$

Using inequalities B1 and B3, inequality B7 is true.

$$\| i_T \|_p \leq \kappa_1 \| i_{oT} \|_p + \kappa_3 \| v_T \|_p + \delta_1 + \delta_3 \qquad (B7)$$

Similarly, one can arrive at a bound for P-norm of v.

$$\| v_T \|_p \leq \kappa_2 \| v_{oT} \|_p + \kappa_4 \| i_T \|_p + \delta_2 + \delta_4 \qquad (B8)$$

B7 and B8 result in inequalities B9 and B10.

$$\| i_T \|_p \leq (1-\kappa_4\kappa_3)^{-1} \{ \kappa_1 \| i_{oT} \|_p + \kappa_3\kappa_2 \| v_{oT} \|_p + \kappa_3(\delta_2+\delta_4) + \delta_1 + \delta_3 \} \qquad (B9)$$

$$\| v_T \|_p \leq (1-\kappa_4\kappa_3)^{-1} \{ \kappa_2 \| v_{oT} \|_p + \kappa_4\kappa_1 \| i_{oT} \|_p + \kappa_4(\delta_1+\delta_3) + \delta_2 + \delta_4 \} \qquad (B10)$$

When $\kappa_3$ and $\kappa_4$ are set to zero then:

$$\| i_T \|_p \leq \kappa_1 \| i_{oT} \|_p + \delta_1 + \delta_3 \qquad (B11)$$

$$\| v_T \|_p \leq \kappa_2 \| v_{oT} \|_p + \delta_2 + \delta_4 \qquad (B12)$$

Inequalities B11 and B12 show that the mapping from $(i_o, v_o) \to (i, v)$ is bounded when $\kappa_3$ and $\kappa_4$ are set to zero. Because this reasoning is valid for every finite T, it follows that i and $v \in L^n_{pe}$.

Next we show that whenever $i_o$ and $v_o$ belong to $L^n_p$ then i and v belong to $L^n_p$ and moreover, the mapping from $(i_o,v_o) \to (i,v)$ is linearly bounded in the sense of Definition 5 (condition b) of Appendix A. The reasoning by which inequalities B11 and B12 were derived still holds when $i_o$ and $v_o$ belong to $L^n_p$. Since the p-norm is a nondecreasing function of T, the subscript T on $i_o$ and $v_o$ can be dropped.

$$\| i_T \|_p \leq \kappa_1 \| i_o \|_p + \delta_1 + \delta_3 \quad \text{(B13)}$$

$$\| v_T \|_p \leq \kappa_2 \| v_o \|_p + \delta_2 + \delta_4 \quad \text{(B14)}$$

Since $i_o$ and $v_o \in L^n_p$, then $\| i_T \|_p$ and $\| v_T \|_p$ in inequalities B13 and B14 are bounded by fixed real numbers. Therefore i and $v \in L^n_p$. Taking the limit values on both sides of inequalities B13 and B14 when $T \to \infty$ it is clear that:

$$\| i \|_p \leq \kappa_1 \| i_o \|_p + \delta_1 + \delta_3 \quad \text{(B12)}$$

$$\| v \|_p \leq \kappa_2 \| v_o \|_p + \delta_2 + \delta_4 \quad \text{(B14)}$$

which proves the linear boundedness of the of the mapping $(i_o,v_o) \to (i,v)$. This shows the $L_p$ stability of the mapping $(i_o,v_o) \to (i,v)$ defined by equations 1 ana 2 under the assumptions of the theorem.

## Appendix C

The objective is to prove how the $L_p$-stability can not be guaranteed for the system represented by equations 1 and 2 when the operator $S_f$ is not $L_p$-stable but causal and the operators $G_{tr}$, $G_f$, and $S_{tr}$ are $L_p$-stable and causal. It is assumed that the operator $S_f: L^n_{pe} \to L^n_{pe}$ and whenever the input belongs to $L^n_p$ the output belongs to $L^n_p$ also. Moreover, the operator $S_f$ is assumed to be nonuniformly linearly bounded in the sense that there exist a finite scalar $\kappa_4(T) \in \mathbb{R}^+$, and a $\delta_4 \in \mathbb{R}$ such that:

$$S_f(i)_T \|_p \leq \kappa_4(T) \| i_T \|_p + \delta_4 \quad \text{(C1)}$$

Note that the operator $S_f$ is not $L_p$-stable since there is no fixed real constant $\kappa_4$ that linearly bounds the mapping $S_f$. Since the operators $G_{tr}$, $G_f$, and, $S_{tr}$ are $L_p$ stable and causal, there exist finite constants $\kappa_1$, $\kappa_2, \kappa_3, \in \mathbb{R}^+$ and $\delta_1$, $\delta_2$, $\delta_3, \in \mathbb{R}$ such that the following inequalities are true. (2)

$$\| G_t(i_o))_T \|_p \leq \kappa_1 \| i_{oT} \|_p + \delta_1 \quad \text{(C2)}$$

$$\| G_f(v_o))_T \|_p \leq \kappa_2 \| v_{oT} \|_p + \delta_2 \quad \text{(C3)}$$

$$\| S_t(v))_T \|_p \leq \kappa_3 \| v_T \|_p + \delta_3 \quad \text{(C4)}$$

where $T \in R^+$

<u>Proof</u>: Considering the system in figure 6 the following equation can be written:

$$v_T = G_f(v_o)_T + S_f(i)_T \quad \text{(C5)}$$

Using the same reasoning as in Appendix B, the following inequality can be derived,

$$\| v_T \|_p \leq (1-\kappa_4(T)\kappa_3)^{-1}(\kappa_2 \|v_{oT}\|_p + \kappa_4(T)\ \kappa_1 \| i_{oT} \|_p + \kappa_4(T)(\delta_1+\delta_3)+\delta_2+\delta_4) \quad \text{(C6)}$$

Defining $\kappa(T)$ as:

$$\kappa(T) = \max\{\kappa_2, \kappa_4(T)\ \kappa_1\} \quad \text{(C7)}$$

inequality C6 can be written as

$$\| v_T \|_p \leq (1-\kappa_4(T)\kappa_3)^{-1}(\kappa(T)\ (\| i_{oT} \|_p + \|v_{oT}\|_p) + \kappa_4(T)(\delta_1+\delta_3)+\delta_2+\delta_4) \quad \text{(C8)}$$

Let's analyze the mapping $(i_o,v_o) \to v$. From inequality C8 it is clear that whenever $i_o$ and $v_o \in L^n_{pe}$ then $v \in L^n_{pe}$. We now show that if $i_o$ and $v_o \in L^n_p$ then $v \in L^n_p$ also. Since $i_o$ and $v_o \in L^n_p$, the value of $\| i_o \|_p + \| v_o \|_p$ and consequently that of $\| i_{oT} \|_p + \| v_{oT} \|_p$ are bounded. Moreover, for given $i_o$ and $v_o$, the values of $\kappa_4(T)$ and $\kappa(T)$ are finite since $S_f: L^n_p \to L^n_p$. Thus, for a given $i_o$ and $v_o$, the bound of $\|v_T\|_p$ in C8 is finite for any finite value of T. This implies that the mapping $(i_o,v_o) \to v$ goes form $L^n_p$ to $L^n_p$. However, this mapping is not linearly bounded in the sense of definition 5 (condition b) of Appendix A because there is no fixed real constant $\kappa$ that could possibly bound the mapping $(i_o,v_o) \to v$. Hence, the system in Figure 6 is not $L_p$-stable under the conditions of the theorem.

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